

Representations of the parafermion vertex operator algebras

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Abstract

The rationality of the parafermion vertex operator algebra $K(\mathfrak{g}, k)$ associated to any finite dimensional simple Lie algebra \mathfrak{g} and any nonnegative integer k is established and the irreducible modules are determined.

1 Introduction

This paper deals with the representation theory of the parafermion vertex operator algebra $K(\mathfrak{g}, k)$ associated to any finite dimensional simple Lie algebra \mathfrak{g} and positive integer k . Using the recent results on abelian orbifolds [6], [42] and coset construction [31], we establish the rationality of $K(\mathfrak{g}, k)$ and determine the irreducible modules. If $\mathfrak{g} = sl_2$, the classification of irreducible modules and rationality were achieved previously in [2]- [3].

The origin of the parafermion vertex operator algebra is the theory of Z -algebra developed in [34, 35, 36] for constructing irreducible highest weight modules for affine Kac-Moody algebra. The main idea was to determine the vacuum space of a module for an affine Kac-Moody algebra. The vacuum space is the space of highest weight vectors for the Heisenberg algebra and is a module for the Z -algebra. Using the language of the conformal field theory and vertex operator algebra, this is the coset theory associated to the simple affine vertex operator algebra $L_{\hat{\mathfrak{g}}}(k, 0)$ and its vertex operator subalgebra $M_{\hat{\mathfrak{h}}}(k)$ generated by the Cartan subalgebra \mathfrak{h} of \mathfrak{g} . That is, the parafermion vertex operator algebra $K(\mathfrak{g}, k)$ is the commutant of $M_{\hat{\mathfrak{h}}}(k)$ in $L_{\hat{\mathfrak{g}}}(k, 0)$ or a coset construction associated to the Lie algebra \mathfrak{g} and its Cartan subalgebra \mathfrak{h} [25].

There have been a lot of investigations of the parafermion vertex operator algebras and parafermion conformal field theory in physics (see [4], [23], [24], [26], [43], [45], [46]). The relation between the parafermion conformal field theory and the Z -algebra has been clarified using the generalized vertex operator algebra in [11]. But a systematic study of the parafermion vertex operator algebra took place only very recently. A set of generators of the parafermion vertex operator algebra $K(\mathfrak{g}, k)$ was determined in [9], [10] and [16]. Moreover, the parafermion vertex operator algebra $K(\mathfrak{g}, k)$ for an arbitrary \mathfrak{g} is generated by $K(\mathfrak{g}^\alpha, k_\alpha)$ where \mathfrak{g}^α is the 3-dimensional subalgebra isomorphic to sl_2 and associated to a root α , and $k_\alpha = \frac{2}{\langle \alpha, \alpha \rangle} k$. From this point of view, $K(sl_2, k)$ is the building block of the general parafermion vertex operator algebra $K(\mathfrak{g}, k)$.

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The representation theory of $K(\mathfrak{g}, k)$ has not been understood well. It is proved in [2] that the Zhu algebra $A(K(sl_2, k))$ is a finite dimensional semisimple associative algebra, the irreducible modules of $K(sl_2, k)$ are exactly those which appear in the integrable highest weight modules of level k for affine Kac-Moody algebra $\widehat{sl_2}$, and $K(\mathfrak{g}, k)$ is C_2 -cofinite for any \mathfrak{g} . Also see [17] on the C_2 -cofiniteness of $K(\mathfrak{g}, k)$.

It is well known that the $L_{\widehat{\mathfrak{g}}}(k, 0)$ is a rational vertex operator algebra [21], [38] and $K(\mathfrak{g}, k)$ is also the commutant of lattice vertex operator algebra $V_{\sqrt{k}Q_L}$ in $L_{\widehat{\mathfrak{g}}}(k, 0)$ where Q_L is the lattice spanned by the long roots [9], [18]. There is a famous conjecture in the coset construction which says that the commutant U^c of a rational vertex operator subalgebra U in a rational vertex operator algebra V is also rational. So it is widely believed that $K(\mathfrak{g}, k)$ should give a new class of rational vertex operator algebras although this can only be proved so far in the case $\mathfrak{g} = sl_2$ and $k \leq 6$ [9].

Here are the main results in this paper: (1) For any simple Lie algebra \mathfrak{g} and any positive integer k , the parafermion vertex operator algebra $K(\mathfrak{g}, k)$ is rational, (2) The irreducible modules of $K(\mathfrak{g}, k)$ are those appeared in the integrable highest weight modules of level k for affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ and we give some identification among these irreducible $K(\mathfrak{g}, k)$ -modules.

Although the parafermion vertex operator algebras $K(\mathfrak{g}, k)$ are subalgebras of the affine vertex operator algebras $L_{\widehat{\mathfrak{g}}}(k, 0)$ their structure are much more complicated. According to [9], $K(sl_2, k)$ is strongly generated by the Virasoro element ω , and highest weight vectors W^3, W^4, W^5 of weights 3, 4, 5. For each root α let $\omega_\alpha, W_\alpha^i$ be the corresponding elements in $K(\mathfrak{g}^\alpha, k_\alpha)$. Then $K(\mathfrak{g}, k)$ is generated by $\omega_\alpha, W_\alpha^i$ for all positive roots α and $i = 3, 4, 5$. Although the Lie bracket $[Y(u, z_1), Y(v, z_2)]$ is complicated for $u, v \in S_\alpha = \{\omega_\alpha, W_\alpha^i | i = 3, 4, 5\}$ but computable [9]. But if α, β are two different positive roots and $u \in S_\alpha, v \in S_\beta$ we do not know the commutator relation $[Y(u, z_1), Y(v, z_2)]$ in general. So it is very difficult to prove the main results directly by using the generators and relations.

The rationality of $K(\mathfrak{g}, k)$ is relatively easy due to recent results in [6], [42]. Let Q_L be the sublattice of root lattice of \mathfrak{g} spanned by the long roots. Then $V_{\sqrt{k}Q_L} \otimes K(\mathfrak{g}, k)$ is a vertex operator subalgebra of $L_{\widehat{\mathfrak{g}}}(k, 0)$. Moreover, $V_{\sqrt{k}Q_L} \otimes K(\mathfrak{g}, k)$ can be realized as fixed point subalgebra $L_{\widehat{\mathfrak{g}}}(k, 0)^G$ for a finite abelian group G . It follows from [6] that $V_{\sqrt{k}Q_L} \otimes K(\mathfrak{g}, k)$ is rational. Using the rationality of $V_{\sqrt{k}Q_L}$, one can easily conclude that $K(\mathfrak{g}, k)$ is rational.

It follows from [31] each irreducible $K(\mathfrak{g}, k)$ -module occurs in an irreducible $\widehat{\mathfrak{g}}$ -module $L_{\widehat{\mathfrak{g}}}(k, \Lambda)$ where Λ is a dominant weight of finite dimensional Lie algebra \mathfrak{g} satisfying $\langle \Lambda, \theta \rangle \leq k$, and θ is the maximal root of \mathfrak{g} . We denote these irreducible modules by $M^{\Lambda, \lambda}$ in this paper where λ lies in $\Lambda + Q$ and Q is the root lattice of \mathfrak{g} . We also find some identification of these irreducible $K(\mathfrak{g}, k)$ -modules by using the simple currents [39] and [41]. It turns out these identifications are complete [1]. The quantum dimensions and the fusion rules have also been determined in [19] and [1].

The connection between the parafermion vertex operator algebras and the commutants of $L_{\widehat{\mathfrak{g}}}(k, 0)$ in $L_{\widehat{\mathfrak{g}}}(1, 0)^{\otimes k}$ has also been investigated in [32], [28], [29].

The paper is organized as follows. We review the vertex operator algebra $V_{\widehat{\mathfrak{g}}}(k, 0)$ associated to affine Kac-Moody algebra $\widehat{\mathfrak{g}}$ and its irreducible quotient $L_{\widehat{\mathfrak{g}}}(k, 0)$ in Section 2. We also discuss the $V_{\widehat{\mathfrak{g}}}(k, 0)$ -modules $V_{\widehat{\mathfrak{g}}}(k, \Lambda)$ generated by any irreducible highest weight \mathfrak{g} -module $L_{\mathfrak{g}}(\Lambda)$. We recall from [16] the vertex operator algebra $N(\mathfrak{g}, k)$ which

is the commutant of the Heisenberg vertex operator algebra $M_{\widehat{\mathfrak{g}}}(k)$ in $V_{\widehat{\mathfrak{g}}}(k, 0)$. We also decompose each $V_{\widehat{\mathfrak{g}}}(k, 0)$ -module $V_{\widehat{\mathfrak{g}}}(k, \Lambda)$ into a direct sum of $M_{\widehat{\mathfrak{h}}}(k) \otimes N(\mathfrak{g}, k)$ -modules. Section 4 is devoted to the study of the parafermion vertex operator algebra $K(\mathfrak{g}, k)$. In particular, $K(\mathfrak{g}, k)$ is the irreducible quotient of $N(\mathfrak{g}, k)$ and is also the commutant of $M_{\widehat{\mathfrak{h}}}(k)$ in $L_{\widehat{\mathfrak{g}}}(k, 0)$. We also decompose each irreducible $L_{\widehat{\mathfrak{g}}}(k, 0)$ -module $L_{\widehat{\mathfrak{g}}}(k, \Lambda)$ into a direct sum of irreducible $M_{\widehat{\mathfrak{h}}}(k) \otimes K(\mathfrak{g}, k)$ -modules $M_{\widehat{\mathfrak{h}}}(k, \lambda) \otimes M^{\Lambda, \lambda}$ for $\lambda \in \Lambda + Q$ where Q is the root lattice of \mathfrak{g} . In addition, we investigate the relation between these irreducible $K(\mathfrak{g}, k)$ -modules $M^{\Lambda, \lambda}$. Both rationality and classification of irreducible modules for $K(\mathfrak{g}, k)$ for any simple Lie algebra \mathfrak{g} are obtained in Section 5.

We assume the readers are familiar with the admissible modules, rationality, $A(V)$ -theory as presented in [12], [13] and [47].

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2 Affine vertex operator algebras

In this section, we recall the basics of vertex operator algebras associated to affine Lie algebras following [21] and [33].

Fix a finite dimensional simple Lie algebra \mathfrak{g} with a Cartan subalgebra \mathfrak{h} . Denote the corresponding root system by Δ and the root lattice by Q . Let $\langle \cdot, \cdot \rangle$ be an invariant symmetric nondegenerate bilinear form on \mathfrak{g} such that $\langle \alpha, \alpha \rangle = 2$ if α is a long root, where we have identified \mathfrak{h} with \mathfrak{h}^* via $\langle \cdot, \cdot \rangle$. We denote the image of $\alpha \in \mathfrak{h}^*$ in \mathfrak{h} by t_α . That is, $\alpha(h) = \langle t_\alpha, h \rangle$ for any $h \in \mathfrak{h}$. Fix simple roots $\{\alpha_1, \dots, \alpha_l\}$ and let Δ_+ be the set of corresponding positive roots. Denote the highest root by θ .

For $\alpha \in \Delta_+$ we denote the root space by \mathfrak{g}_α and fix $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ and $h_\alpha = \frac{2}{\langle \alpha, \alpha \rangle} t_\alpha \in \mathfrak{h}$ such that $[x_\alpha, x_{-\alpha}] = h_\alpha$, $[h_\alpha, x_{\pm\alpha}] = \pm 2x_{\pm\alpha}$. That is, $\mathfrak{g}^\alpha = \mathbb{C}x_\alpha + \mathbb{C}h_\alpha + \mathbb{C}x_{-\alpha}$ is isomorphic to sl_2 by sending x_α to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $x_{-\alpha}$ to $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and h_α to $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\langle h_\alpha, h_\alpha \rangle = 2 \frac{\langle \theta, \theta \rangle}{\langle \alpha, \alpha \rangle}$ and $\langle x_\alpha, x_{-\alpha} \rangle = \frac{\langle \theta, \theta \rangle}{\langle \alpha, \alpha \rangle}$ for all $\alpha \in \Delta$.

Recall that $\widehat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ is the affine Lie algebra associated to \mathfrak{g} with Lie bracket

$$[a(m), b(n)] = [a, b](m+n) + m\langle a, b \rangle \delta_{m+n, 0} K, [K, \widehat{\mathfrak{g}}] = 0$$

for $a, b \in \mathfrak{g}$ and $m, n \in \mathbb{Z}$ where $a(m) = a \otimes t^m$. Note that $\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}K$ is a subalgebra of $\widehat{\mathfrak{g}}$.

Fix a positive integer k and a weight $\Lambda \in \mathfrak{h}^*$. Let $L_{\mathfrak{g}}(\Lambda)$ be the irreducible highest weight module for \mathfrak{g} with highest weight Λ and

$$V_{\widehat{\mathfrak{g}}}(k, \Lambda) = Ind_{\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}K}^{\widehat{\mathfrak{g}}} L_{\mathfrak{g}}(\Lambda)$$

be the induced $\widehat{\mathfrak{g}}$ -module where $\mathfrak{g} \otimes \mathbb{C}[t]t$ acts as 0, $\mathfrak{g} = \mathfrak{g} \otimes t^0$ acts as \mathfrak{g} and K acts as k on $L_{\mathfrak{g}}(\Lambda)$. Then $V_{\widehat{\mathfrak{g}}}(k, \Lambda)$ has a unique maximal submodule $\mathcal{J}(k, \Lambda)$ and we denote the irreducible quotient by $L_{\widehat{\mathfrak{g}}}(k, \Lambda)$. In the case $\Lambda = 0$, the maximal submodule $\mathcal{J} = \mathcal{J}(k, 0)$ of $V_{\widehat{\mathfrak{g}}}(k, 0)$ is generated by $x_\theta(-1)^{k+1}1$ [30] where $1 = 1 \otimes 1 \in V_{\widehat{\mathfrak{g}}}(k, 0)$. The \mathcal{J} is also generated by $x_{-\theta}(0)^{k+1}x_\theta(-1)^{k+1}1$ [10], [16]. Moreover, $L_{\widehat{\mathfrak{g}}}(k, \Lambda)$ is integrable if and only if Λ is a dominant weight such that $\langle \Lambda, \theta \rangle \leq k$ [30]. Denote the set of such Λ by P_+^k .

It is well known that $V_{\widehat{\mathfrak{g}}}(k, 0)$ is a vertex operator algebra generated by $a(-1)\mathbf{1}$ for $a \in \mathfrak{g}$ such that

$$Y(a(-1)\mathbf{1}, z) = a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}$$

with the vacuum vector $\mathbf{1} = 1$ and the Virasoro vector

$$\omega_{\text{aff}} = \frac{1}{2(k+h^\vee)} \left(\sum_{i=1}^l u_i(-1)u_i(-1)\mathbf{1} + \sum_{\alpha \in \Delta} \frac{\langle \alpha, \alpha \rangle}{2} x_\alpha(-1)x_{-\alpha}(-1)\mathbf{1} \right)$$

of central charge $\frac{k \dim \mathfrak{g}}{k+h^\vee}$ (see [21], [33]), where h^\vee is the dual Coxeter number of \mathfrak{g} and $\{u_i | i = 1, \dots, l\}$ is an orthonormal basis of \mathfrak{h} . Moreover, each $V_{\widehat{\mathfrak{g}}}(k, \Lambda)$ is a module for $V_{\widehat{\mathfrak{g}}}(k, 0)$ for any Λ [21], [33]. As usual, we let $Y(\omega_{\text{aff}}, z) = \sum_{n \in \mathbb{Z}} L_{\text{aff}}(n)z^{-n-2}$. Then

$$V_{\widehat{\mathfrak{g}}}(k, \Lambda) = \bigoplus_{n \geq 0} V_{\widehat{\mathfrak{g}}}(k, \Lambda)_{n_\Lambda + n}$$

where $V_{\widehat{\mathfrak{g}}}(k, \Lambda)_{n_\Lambda + n} = \{v \in V_{\widehat{\mathfrak{g}}}(k, \Lambda) | L_{\text{aff}}(0)v = (n_\Lambda + n)v\}$, $n_\Lambda = \frac{(\Lambda, \Lambda + 2\rho)}{2(k+h^\vee)}$ and $\rho = \frac{1}{2} \sum_{\alpha \in \Delta_+} \alpha$. The top level $V_{\widehat{\mathfrak{g}}}(k, \Lambda)_{n_\Lambda}$ of $V_{\widehat{\mathfrak{g}}}(k, \Lambda)$ is exactly the $L_{\mathfrak{g}}(\Lambda)$.

Now \mathcal{J} is the maximal ideal of $V_{\widehat{\mathfrak{g}}}(k, 0)$ and $L_{\widehat{\mathfrak{g}}}(k, 0)$ is a simple, rational vertex operator algebra such that the irreducible $L_{\widehat{\mathfrak{g}}}(k, 0)$ -modules are exactly the integrable highest weight $\widehat{\mathfrak{g}}$ -modules $L_{\widehat{\mathfrak{g}}}(K, \Lambda)$ of level k for $\Lambda \in P_+^k$ (cf. [11], [21], [33]).

3 Vertex operator algebras $N(\mathfrak{g}, k)$

We recall from [9], [10], [11], [16] the construction of vertex operator algebras $N(\mathfrak{g}, k)$ and their generators.

Let $M_{\widehat{\mathfrak{h}}}(k)$ be the vertex operator subalgebra of $V_{\widehat{\mathfrak{g}}}(k, 0)$ generated by $h(-1)\mathbf{1}$ for $h \in \mathfrak{h}$ with the Virasoro element

$$\omega_{\mathfrak{h}} = \frac{1}{2k} \sum_{i=1}^l u_i(-1)u_i(-1)\mathbf{1}$$

of central charge l . For $\lambda \in \mathfrak{h}^*$, denote by $M_{\widehat{\mathfrak{h}}}(k, \lambda)$ the irreducible highest weight module for $\widehat{\mathfrak{h}}$ with a highest weight vector e^λ such that $h(0)e^\lambda = \lambda(h)e^\lambda$ for $h \in \mathfrak{h}$.

Let $N(\mathfrak{g}, k) = \{v \in V_{\widehat{\mathfrak{g}}}(k, 0) | u_n v = 0, u \in M_{\widehat{\mathfrak{h}}}(k), n \geq 0\}$ be the commutant [21] of $M_{\widehat{\mathfrak{h}}}(k)$ in $V_{\widehat{\mathfrak{g}}}(k, 0)$. It is easy to see that $N(\mathfrak{g}, k) = \{v \in V_{\widehat{\mathfrak{g}}}(k, 0) | h(n)v = 0, h \in \mathfrak{h}, n \geq 0\}$. The $N(\mathfrak{g}, k)$ is a vertex operator algebra with the Virasoro vector $\omega = \omega_{\text{aff}} - \omega_{\mathfrak{h}}$ whose central charge is $\frac{k \dim \mathfrak{g}}{k+h^\vee} - l$.

For $\alpha \in \Delta_+$, let $k_\alpha = \frac{\langle \theta, \theta \rangle}{\langle \alpha, \alpha \rangle} k$. Clearly, k_α is a nonnegative integer. Set

$$\omega_\alpha = \frac{1}{2k_\alpha(k_\alpha + 2)} (-k_\alpha h_\alpha(-2)\mathbf{1} - h_\alpha(-1)^2\mathbf{1} + 2k_\alpha x_\alpha(-1)x_{-\alpha}(-1)\mathbf{1}), \quad (3.1)$$

$$\begin{aligned} W_\alpha^3 = & k_\alpha^2 h_\alpha(-3)\mathbf{1} + 3k_\alpha h_\alpha(-2)h_\alpha(-1)\mathbf{1} + 2h_\alpha(-1)^3\mathbf{1} \\ & - 6k_\alpha h_\alpha(-1)x_\alpha(-1)x_{-\alpha}(-1)\mathbf{1} + 3k_\alpha^2 x_\alpha(-2)x_{-\alpha}(-1)\mathbf{1} - 3k_\alpha^2 x_\alpha(-1)x_{-\alpha}(-2)\mathbf{1}. \end{aligned} \quad (3.2)$$

One can also see [9], [16] for the definition of the highest vector W_α^4, W_α^5 of weights 4 and 5. Then $N(\mathfrak{g}, k)$ is generated by $\omega_\alpha, W_\alpha^3$ for $\alpha \in \Delta_+$ [16] and the subalgebra of $N(\mathfrak{g}, k)$ generated by ω_α and W_α^3 for fixed α is isomorphic to $N(sl_2, k_\alpha)$ (cf. [9], [10], [16]).

Remark 3.1. It is proved in [9] that $N(sl_2, k)$ is strongly generated by ω, W^3, W^4, W^5 where we omit the α as there is only one positive root. But it is not clear if $N(\mathfrak{g}, k)$ is strongly generated by $\omega_\alpha, W_\alpha^i$ for $\alpha \in \Delta_+$ and $i = 3, 4, 5$. We also do not know the commutator relation $[Y(u, z_1), Y(v, z_2)]$ in general for these generators. It is definitely important to find the relation $[Y(u, z_1), Y(v, z_2)]$ for the generators explicitly. This will help to understand the structure of $N(\mathfrak{g}, k)$ better.

For $\Lambda, \lambda \in \mathfrak{h}^*$, set

$$V_{\widehat{\mathfrak{g}}}(k, \Lambda)(\lambda) = \{v \in V_{\widehat{\mathfrak{g}}}(k, \Lambda) | h(0)v = \lambda(h)v, \forall h \in \mathfrak{h}\}.$$

Then we have

$$V_{\widehat{\mathfrak{g}}}(k, \Lambda) = \bigoplus_{\lambda \in Q + \Lambda} V_{\widehat{\mathfrak{g}}}(k, \Lambda)(\lambda). \quad (3.3)$$

Note that $M_{\widehat{\mathfrak{h}}}(k) \otimes N(\mathfrak{g}, k)$ is a vertex operator subalgebra of $V_{\widehat{\mathfrak{g}}}(k, 0)$ isomorphic to $V_{\widehat{\mathfrak{g}}}(k, 0)(0)$ and $V_{\widehat{\mathfrak{g}}}(k, \Lambda)(\lambda) = M_{\widehat{\mathfrak{h}}}(k, \lambda) \otimes N^{\Lambda, \lambda}$ as a module for $M_{\widehat{\mathfrak{h}}}(k) \otimes N(\mathfrak{g}, k)$ where

$$N^{\Lambda, \lambda} = \{v \in V_{\widehat{\mathfrak{g}}}(k, \Lambda) | h(m)v = \lambda(h)\delta_{m,0}v \text{ for } h \in \mathfrak{h}, m \geq 0\}$$

is the space of highest weight vectors with highest weight λ for $\widehat{\mathfrak{h}}$. Clearly, $N(\mathfrak{g}, k) = N^{0,0}$.

For any $\alpha \in \Delta$, the subalgebra of $N(\mathfrak{g}, k)$ generated by $\omega_\alpha, W_\alpha^3$ is isomorphic to $N(\mathfrak{g}^\alpha, k_\alpha)$ and the subalgebra of $V_{\widehat{\mathfrak{g}}}(k, 0)(0)$ generated by $\omega_\alpha, W_\alpha^3, t_\alpha(-1)\mathbf{1}$ is isomorphic to $V_{\widehat{\mathfrak{g}^\alpha}}(k_\alpha, 0)(0)$ [16]. Consequently, we regard $N(\mathfrak{g}, k_\alpha)$ as a subalgebra of $N(\mathfrak{g}, k)$, and $V_{\widehat{\mathfrak{g}^\alpha}}(k_\alpha, 0)(0)$ as a subalgebra of $V_{\widehat{\mathfrak{g}}}(k, 0)(0)$. Here is a stronger result on the generators of $N(\mathfrak{g}, k)$. Recall that $\{\alpha_1, \dots, \alpha_l\}$ are the simple roots.

Proposition 3.2. The vertex operator algebra $N(\mathfrak{g}, k)$ is generated by $N(\mathfrak{g}^{\alpha_i}, k_{\alpha_i})$ or $\omega_{\alpha_i}, W_{\alpha_i}^3$ for $i = 1, \dots, l$.

Proof. From [16] we know that the vertex operator subalgebra $V_{\widehat{\mathfrak{g}}}(k, 0)(0) = M_{\widehat{\mathfrak{h}}}(k) \otimes N(\mathfrak{g}, k)$ is generated by $t_\alpha(-1)\mathbf{1}$ and $x_{-\alpha}(-2)x_\alpha(-1)\mathbf{1}$ for $\alpha \in \Delta_+$. We first prove that $V_{\widehat{\mathfrak{g}}}(k, 0)(0)$ is generated by $t_{\alpha_i}(-1)\mathbf{1}$ and $x_{-\alpha_i}(-2)x_{\alpha_i}(-1)\mathbf{1}$ for $1 \leq i \leq l$. In other words, we only need the simple roots for the generators.

Since every positive root can be generated from simple roots we see that $V_{\widehat{\mathfrak{g}}}(k_\alpha, 0)(0)$ is spanned by

$$a_1(-m_1) \cdots a_s(-m_s)x_{\beta_1}(-n_1)x_{\beta_2}(-n_2) \cdots x_{\beta_t}(-n_t)\mathbf{1}$$

where $a_i \in \mathfrak{h}, \beta_j \in \{\pm\alpha_1, \dots, \pm\alpha_l\}, m_i > 0, n_j \geq 0$ and $\beta_1 + \beta_2 + \cdots + \beta_t = 0$. By Proposition 4.5.8 of [33] we see that $V_{\widehat{\mathfrak{g}}}(k, 0)(0)$ is spanned by

$$a_1(-m_1) \cdots a_s(-m_s)x_{\beta_1}(n_1)x_{-\beta_1}(p_1)x_{\beta_2}(n_2)x_{-\beta_2}(p_2) \cdots x_{\beta_t}(n_t)x_{-\beta_t}(p_t)\mathbf{1}$$

where a_i, m_i are as before and $\beta_j \in \{\alpha_1, \dots, \alpha_l\}, n_j, p_j \in \mathbb{Z}$. It follows from Proposition 4.5.7 [33] that $V_{\widehat{\mathfrak{g}}}(k, 0)(0)$ is spanned by

$$a_1(-m_1) \cdots a_s(-m_s)u_{n_1}^1 \cdots u_{n_t}^t \mathbf{1}$$

where $u^j \in V_{\widehat{\mathfrak{g}^{\beta_j}}}(k_{\beta_j}, 0)(0), \beta_i \in \{\alpha_1, \dots, \alpha_l\}$, and $n_j \in \mathbb{Z}$.

Since each $V_{\widehat{\mathfrak{g}^{\alpha_i}}}(k_{\alpha_i}, 0)(0)$ is generated by $t_{\alpha_i}(-1)$ and $\omega_{\alpha_i}, W_{\alpha_i}^3$ we see immediately that $N(\mathfrak{g}, k)$ is generated by $\omega_{\alpha_i}, W_{\alpha_i}^3$ or $N(\mathfrak{g}^{\alpha_i}, k_{\alpha_i})$ for $i = 1, \dots, l$. The proof is complete. \square

Next we discuss the decomposition of $N^{\Lambda, \lambda}$ into the direct sum of weight spaces. Consider the weight space decomposition $L_{\mathfrak{g}}(\Lambda) = \bigoplus_{\lambda \in \mathfrak{h}^*} L_{\mathfrak{g}}(\Lambda)_{\lambda}$ where $L_{\mathfrak{g}}(\Lambda)_{\lambda}$ is the weight space of $L_{\mathfrak{g}}(\Lambda)$ with weight λ . Let $P(L_{\mathfrak{g}}(\Lambda)) = \{\lambda \in \mathfrak{h}^* | L_{\mathfrak{g}}(\Lambda)_{\lambda} \neq 0\}$ be the weights of $L_{\mathfrak{g}}(\Lambda)$.

Lemma 3.3. *Let $\lambda \in P(L_{\mathfrak{g}}(\Lambda))$. Then the $N(\mathfrak{g}, k)$ -module*

$$N^{\Lambda, \lambda} = \bigoplus_{n \geq 0} N^{\Lambda, \lambda}_{n_{\Lambda} - \frac{\langle \lambda, \lambda \rangle}{2k} + n} \quad (3.4)$$

is generated by the irreducible $A(N(\mathfrak{g}, k))$ -module $N^{\Lambda, \lambda}_{n_{\Lambda} - \frac{\langle \lambda, \lambda \rangle}{2k}} = L_{\mathfrak{g}}(\Lambda)_{\lambda}$ where

$$N^{\Lambda, \lambda}_{n_{\Lambda} - \frac{\langle \lambda, \lambda \rangle}{2k} + n} = \{w \in N^{\Lambda, \lambda} | L(0)w = (n_{\Lambda} - \frac{\langle \lambda, \lambda \rangle}{2k} + n)w\}$$

and $Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2}$.

Proof. Write $Y(\omega_{\mathfrak{h}}, z) = \bigoplus_{n \in \mathbb{Z}} L_{\mathfrak{h}}(n)z^{-n-2}$ and note that $L_{\mathfrak{h}}(0)$ acts on $L_{\mathfrak{g}}(\Lambda)_{\lambda}$ as $\frac{\langle \lambda, \lambda \rangle}{2k}$. The decomposition (3.4) follows immediately.

Next we prove that $N^{\Lambda, \lambda}$ is generated by $L_{\mathfrak{g}}(\Lambda)_{\lambda}$ which is an irreducible $A(N(\mathfrak{g}, k))$ -module for $\lambda \in P(L_{\mathfrak{g}}(\Lambda))$. We observe that $V_{\widehat{\mathfrak{g}}}(k, \Lambda)$ is generated by any nonzero vector in $L_{\mathfrak{g}}(\Lambda)$ from the construction of $V_{\widehat{\mathfrak{g}}}(k, \Lambda)$ as $L_{\mathfrak{g}}(\Lambda)$ is an irreducible \mathfrak{g} -module. It follows from [14] and [37] that for any nonzero vector $w \in L_{\mathfrak{g}}(\Lambda)_{\lambda}$, $V_{\widehat{\mathfrak{g}}}(k, \Lambda)$ is spanned by $u_n w$ for $u \in V_{\widehat{\mathfrak{g}}}(k, 0)$ and $n \in \mathbb{Z}$. Since $u_n w \in V_{\widehat{\mathfrak{g}}}(k, \Lambda)(\alpha + \lambda)$ for $u \in V_{\widehat{\mathfrak{g}}}(k, 0)(\alpha)$ for $\alpha \in Q$, $V_{\widehat{\mathfrak{g}}}(k, \Lambda)(\lambda)$ is spanned by $u_n w$ for $u \in M_{\widehat{\mathfrak{h}}}(k) \otimes N(\mathfrak{g}, k)$ and $n \in \mathbb{Z}$. It follows that $N^{\Lambda, \lambda}$ is spanned by $u_n w$ for $u \in N(\mathfrak{g}, k)$ and $n \in \mathbb{Z}$. This implies that $N^{\Lambda, \lambda}$ is generated by $L_{\mathfrak{g}}(\Lambda)_{\lambda}$ and $L_{\mathfrak{g}}(\Lambda)_{\lambda}$ is a simple $A(M_{\widehat{\mathfrak{h}}}(k) \otimes N(\mathfrak{g}, k))$ -module. As $A(M_{\widehat{\mathfrak{h}}}(k) \otimes N(\mathfrak{g}, k))$ is isomorphic to $A(M_{\widehat{\mathfrak{h}}}(k)) \otimes_{\mathbb{C}} A(N(\mathfrak{g}, k))$ [15] and $A(M_{\widehat{\mathfrak{h}}}(k))$ is commutative, we conclude that $L_{\mathfrak{g}}(\Lambda)_{\lambda}$ is an irreducible $A(N(\mathfrak{g}, k))$ -module. \square

The universal enveloping algebra $U(\widehat{\mathfrak{g}})$ is Q -graded:

$$U(\widehat{\mathfrak{g}}) = \bigoplus_{\alpha \in Q} U(\widehat{\mathfrak{g}})(\alpha) \quad (3.5)$$

where $U(\widehat{\mathfrak{g}})(\alpha) = \{v \in U(\widehat{\mathfrak{g}}) | [h(0), v] = \alpha(h)v, \forall h \in \mathfrak{h}\}$. Then $U(\widehat{\mathfrak{g}})(0)$ is an associative subalgebra of $U(\widehat{\mathfrak{g}})$ and is generated by $K, h(m), x_{\alpha_1}(s_1) \cdots x_{\alpha_n}(s_n)$ for $h \in \mathfrak{h}, \alpha_i \in \Delta$ with $\sum_i \alpha_i = 0$ and $m, s_i \in \mathbb{Z}$. Note that $U(\widehat{\mathfrak{g}}) = \bigoplus_{n \in \mathbb{Z}} U(\widehat{\mathfrak{g}})_n$ is also \mathbb{Z} -graded such that $\deg x(n) = -n$ for $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$, and $\deg K = 0$. This induces a \mathbb{Z} -gradation on $U(\widehat{\mathfrak{g}})(0)$. It is clear that for $\Lambda \in P_+^k$ and $\lambda \in P(L_{\mathfrak{g}}(\Lambda))$, $V_{\widehat{\mathfrak{g}}}(k, \Lambda)(\lambda) = M_{\widehat{\mathfrak{h}}}(k, \lambda) \otimes N^{\Lambda, \lambda}$ is a $U(\widehat{\mathfrak{g}})(0)$ -module and $V_{\widehat{\mathfrak{g}}}(k, \Lambda)(\lambda)_{n_{\Lambda}} = L_{\mathfrak{g}}(\Lambda)_{\lambda}$ is an irreducible $U(\widehat{\mathfrak{g}})(0)_0$ -module where $V_{\widehat{\mathfrak{g}}}(k, \Lambda)(\lambda)_{n_{\Lambda}} = V_{\widehat{\mathfrak{g}}}(k, \Lambda)(\lambda) \cap V_{\widehat{\mathfrak{g}}}(k, \Lambda)_{n_{\Lambda}}$. The following Lemma is immediate now.

Lemma 3.4. *Let $\Lambda \in P_+^k$ and $\lambda \in P(L_{\mathfrak{g}}(\Lambda))$. Then $V_{\widehat{\mathfrak{g}}}(k, \Lambda)(\lambda) = M_{\widehat{\mathfrak{h}}}(k, \lambda) \otimes N^{\Lambda, \lambda}$ is a Verma $U(\widehat{\mathfrak{g}})(0)$ -module generated by $e^{\lambda} \otimes L_{\mathfrak{g}}(\Lambda)_{\lambda}$ in the sense that $Ie^{\lambda} \otimes L_{\mathfrak{g}}(\Lambda)_{\lambda} = 0$ where I is the left ideal of $U(\widehat{\mathfrak{g}})(0)$ generated by $h(n), x_{\alpha_1}(s_1) \cdots x_{\alpha_p}(s_p)$ for $h \in \mathfrak{h}, \alpha_i \in \Delta$ with $\sum_i \alpha_i = 0, n, s_p > 0$, and any $U(\widehat{\mathfrak{g}})(0)$ module W generated by $e^{\lambda} \otimes L_{\mathfrak{g}}(\Lambda)_{\lambda}$ with $Ie^{\lambda} \otimes L_{\mathfrak{g}}(\Lambda)_{\lambda} = 0$ is a quotient of $V_{\widehat{\mathfrak{g}}}(k, \Lambda)(\lambda)$.*

We next say few words on certain Verma type modules for vertex operator algebra $N(\mathfrak{g}, k)$. Let V be a vertex operator algebra and $A(V)$ be its Zhu algebra. Let U be a simple $A(V)$ -module. Recall from [13] and [8] that the Verma type admissible V -module $M(U) = \bigoplus_{n \geq 0} M(U)(n)$ generated by $M(U)(0) = U$ has the property: any admissible V -module $W = \bigoplus_{n \geq 0} W(n)$ generated by $W(0) = U$ is a quotient of $M(U)$. We firmly believe that $N^{\Lambda, \lambda}$ is a Verma type admissible module for $N(\mathfrak{g}, k)$ generated by $L_{\widehat{\mathfrak{g}}}(\Lambda)_\lambda$ for $\Lambda \in P_+^k$ and $\lambda \in P(L_{\widehat{\mathfrak{g}}}(\Lambda))$. Unfortunately we cannot prove this result in the paper.

4 Vertex operator algebras $K(\mathfrak{g}, k)$

We study the parafermion vertex operator algebra $K(\mathfrak{g}, k)$ in this section. We get a list of irreducible modules $M^{\Lambda, \lambda}$ from the irreducible $L_{\widehat{\mathfrak{g}}}(k, 0)$ -modules $L_{\widehat{\mathfrak{g}}}(k, \Lambda)$ for $\Lambda \in P_+^k$ and $\lambda \in \Lambda + Q$. We also discuss how to use the lattice vertex operator subalgebra of $L_{\widehat{\mathfrak{g}}}(k, 0)$ and the simple currents of $L_{\widehat{\mathfrak{g}}}(k, 0)$ to give identifications between different $M^{\Lambda, \lambda}$.

Recall the irreducible quotient $L_{\widehat{\mathfrak{g}}}(k, \Lambda)$ of $V_{\widehat{\mathfrak{g}}}(k, \Lambda)$ for $\Lambda \in P_+^k$. Again, the Heisenberg vertex operator algebra $M_{\widehat{\mathfrak{h}}}(k)$ generated by $h(-1)\mathbf{1}$ for $h \in \mathfrak{h}$ is a simple subalgebra of $L_{\widehat{\mathfrak{g}}}(k, 0)$ and $L_{\widehat{\mathfrak{g}}}(k, \Lambda)$ is a completely reducible $M_{\widehat{\mathfrak{h}}}(k)$ -module. We have a decomposition

$$L_{\widehat{\mathfrak{g}}}(k, \Lambda) = \bigoplus_{\lambda \in Q + \Lambda} M_{\widehat{\mathfrak{h}}}(k, \lambda) \otimes M^{\Lambda, \lambda} \quad (4.1)$$

as modules for $M_{\widehat{\mathfrak{h}}}(k)$, where

$$M^{\Lambda, \lambda} = \{v \in L_{\widehat{\mathfrak{g}}}(k, \Lambda) \mid h(m)v = \lambda(h)\delta_{m,0}v \text{ for } h \in \mathfrak{h}, m \geq 0\}.$$

Moreover, $M^{\Lambda, \lambda} = N^{\Lambda, \lambda} / (N^{\Lambda, \lambda} \cap \mathcal{J}(k, \Lambda))$. Recall (3.4). We also can write decomposition (4.1) as

$$L_{\widehat{\mathfrak{g}}}(k, \Lambda) = \bigoplus_{\lambda \in Q + \Lambda} L_{\widehat{\mathfrak{g}}}(k, \Lambda)(\lambda)$$

where $L_{\widehat{\mathfrak{g}}}(k, \Lambda)(\lambda) = M_{\widehat{\mathfrak{h}}}(k, \lambda) \otimes M^{\Lambda, \lambda}$ is the weight space of $L_{\widehat{\mathfrak{g}}}(k, \Lambda)$ for Lie algebra \mathfrak{g} with weight λ . Moreover, each $L_{\widehat{\mathfrak{g}}}(k, \Lambda)(\lambda)$ is a module for $U(\widehat{\mathfrak{g}})(0)$.

Set $K(\mathfrak{g}, k) = M^{0,0}$. Then $K(\mathfrak{g}, k)$ is the commutant of $M_{\widehat{\mathfrak{h}}}(k)$ in $L_{\widehat{\mathfrak{g}}}(k, 0)$ and is called the parafermion vertex operator algebra associated to the irreducible highest weight module $L_{\widehat{\mathfrak{g}}}(k, 0)$ for $\widehat{\mathfrak{g}}$. The Virasoro vector of $K(\mathfrak{g}, k)$ is given by

$$\omega = \omega_{\text{aff}} - \omega_{\mathfrak{h}},$$

where we still use $\omega_{\text{aff}}, \omega_{\mathfrak{h}}$ to denote their images in $L_{\widehat{\mathfrak{g}}}(k, 0)$. Since the Virasoro vector of $M_{\widehat{\mathfrak{h}}}(k)$ is $\omega_{\mathfrak{h}}$, we have

$$K(\mathfrak{g}, k) = \{v \in L_{\widehat{\mathfrak{g}}}(k, 0) \mid (\omega_{\mathfrak{h}})_0 v = 0\}.$$

Note that $K(\mathfrak{g}, k)$ is a quotient of $N(\mathfrak{g}, k)$. We still denote by $\omega_\alpha, W_\alpha^3$ for their images in $K(\mathfrak{g}, k)$. It follows from [16] that the subalgebra of $K(\mathfrak{g}, k)$ generated by $\omega_\alpha, W_\alpha^3$ is isomorphic to $K(\mathfrak{g}^\alpha, k_\alpha)$. So we can regard $K(\mathfrak{g}^\alpha, k_\alpha)$ as subalgebra of $K(\mathfrak{g}, k)$. The following result is an immediate consequence of Proposition 3.2

Proposition 4.1. *The vertex operator algebra $K(\mathfrak{g}, k)$ is generated by $K(\mathfrak{g}^{\alpha_i}, k_{\alpha_i})$ or $\omega_{\alpha_i}, W_{\alpha_i}^3$ for $i = 1, \dots, l$.*

We now turn our attention to the irreducible $K(\mathfrak{g}, k)$ -modules.

Lemma 4.2. *Let $\Lambda \in P_+^k$ and $\lambda \in \Lambda + Q$. Then $M^{\Lambda, \lambda}$ is an irreducible $K(\mathfrak{g}, k)$ -module.*

Proof. Since $M_{\hat{\mathfrak{h}}}(k) \otimes K(\mathfrak{g}, k) = L_{\hat{\mathfrak{g}}}(k, 0)(0)$ and $M_{\hat{\mathfrak{h}}}(k, \lambda)$ is an irreducible $M_{\hat{\mathfrak{h}}}(k)$ -module, it is good enough to show that $M_{\hat{\mathfrak{h}}}(k, \lambda) \otimes M^{\Lambda, \lambda} = L_{\hat{\mathfrak{g}}}(k, \Lambda)(\lambda)$ is an irreducible $L_{\hat{\mathfrak{g}}}(k, 0)(0)$ -module. Note that $L_{\hat{\mathfrak{g}}}(k, \Lambda)$ is an irreducible $L_{\hat{\mathfrak{g}}}(k, 0)$ -module. It follows from [14], [33], [37] that for any nonzero $w \in L_{\hat{\mathfrak{g}}}(k, \Lambda)(\lambda)$, $L_{\hat{\mathfrak{g}}}(k, \Lambda)$ is spanned by $u_n w$ for $u \in L_{\hat{\mathfrak{g}}}(k, 0)(\alpha)$ with $\alpha \in Q$, and $n \in \mathbb{Z}$. Clearly, $u_n w \in L_{\hat{\mathfrak{g}}}(k, \Lambda)(\lambda + \alpha)$. This implies that $L_{\hat{\mathfrak{g}}}(k, \Lambda)(\lambda)$ is spanned by $u_n w$ for $u \in L_{\hat{\mathfrak{g}}}(k, 0)(0)$ and $n \in \mathbb{Z}$. That is, $L_{\hat{\mathfrak{g}}}(k, \Lambda)(\lambda)$ is an irreducible $L_{\hat{\mathfrak{g}}}(k, 0)(0)$ -module. \square

But not all these irreducible modules $M^{\Lambda, \lambda}$ are different. We give some identifications of these modules using lattice vertex operator algebras of $L_{\hat{\mathfrak{g}}}(k, 0)$ and simple currents of $L_{\hat{\mathfrak{g}}}(k, 0)$.

Recall the root lattice Q of \mathfrak{g} . Let Q_L be the sublattice of Q spanned by the long roots. Then Q and Q_L have the same rank. It is known from [18], [30] that the lattice vertex operator algebra $V_{\sqrt{k}Q_L}$ (see [5] and [20]) is a subalgebra of $L_{\hat{\mathfrak{g}}}(k, 0)$. Moreover, $M_{\hat{\mathfrak{h}}}(k)$ is a subalgebra of $V_{\sqrt{k}Q_L}$ with the same Virasoro element. In fact,

$$V_{\sqrt{k}Q_L} = M_{\hat{\mathfrak{h}}}(k) \otimes \mathbb{C}[\sqrt{k}Q_L] = \bigoplus_{\alpha \in Q_L} M_{\hat{\mathfrak{h}}}(k, \sqrt{k}\alpha)$$

as a module for $M_{\hat{\mathfrak{h}}}(k)$ (see (4.1)). It is important to point out that in the definition of vertex operator algebra $M_{\hat{\mathfrak{h}}}(k)$ we use the bilinear form \langle, \rangle . If we use the standard notation for the lattice vertex operator algebra $V_{\sqrt{k}Q_L} = M(1) \otimes \mathbb{C}[\sqrt{k}Q_L]$ we have to use another bilinear form $(,) = k\langle, \rangle$. The actions of $h(0)$ on $V_{\sqrt{k}Q_L}$ -modules for $h \in \mathfrak{h}$ also uses $(,)$ instead of \langle, \rangle .

Remark 4.3. *The parafermion vertex operator algebra $K(\mathfrak{g}, k)$ is also the commutant of the rational vertex operator algebra $V_{\sqrt{k}Q_L}$ [7], [12] in rational vertex operator algebra $L_{\hat{\mathfrak{g}}}(k, 0)$. This explains why one expects that $K(\mathfrak{g}, k)$ is rational.*

Let L be an even lattice. As usual, we denote the dual lattice of L by L° . It is known from [7] that the irreducible V_L -modules are given by $V_{L+\lambda}$ where $\lambda \in L^\circ$. Moreover, $V_{L+\lambda} = V_{L+\mu}$ if $\lambda - \mu \in L$. Consider the coset decomposition $Q = \cup_{i \in Q/kQ_L} (kQ_L + \beta_i)$. Since V_L is rational for any positive definite even lattice L (see [7], [12]), we have the decomposition

$$L_{\hat{\mathfrak{g}}}(k, \Lambda) = \bigoplus_{i \in Q/kQ_L} V_{\sqrt{k}Q_L + \frac{1}{\sqrt{k}}(\Lambda + \beta_i)} \otimes M^{\Lambda, \Lambda + \beta_i} \quad (4.2)$$

as modules for $V_{\sqrt{k}Q_L} \otimes K(\mathfrak{g}, k)$ where $M^{\Lambda, \lambda}$ is as before. Again, as a $M_{\hat{\mathfrak{h}}}(k)$ -module

$$V_{\sqrt{k}Q_L + \frac{1}{\sqrt{k}}(\Lambda + \alpha)} = \bigoplus_{\beta \in Q_L} M_{\hat{\mathfrak{h}}}(k, k\beta + \Lambda + \alpha). \quad (4.3)$$

That is, for $h \in \mathfrak{h}$, $h(0)$ acts on $V_{\sqrt{k}Q_L} \otimes K(\mathfrak{g}, k)$ -module $V_{\sqrt{k}Q_L + \frac{1}{\sqrt{k}}\lambda} \otimes M^{\Lambda, \lambda}$ is given by

$$h(0)(u \otimes e^{\sqrt{k}\beta + \frac{1}{\sqrt{k}}\lambda}) \otimes w = \langle h, k\beta + \lambda \rangle (u \otimes e^{\sqrt{k}\beta + \frac{1}{\sqrt{k}}\lambda}) \otimes w$$

for $\beta \in Q_L$, $\lambda \in \Lambda + Q$, $u \in M_{\hat{\mathfrak{h}}}(k)$ and $w \in M^{\Lambda, \lambda}$.

Here is our first identification among $M^{\Lambda, \lambda}$. In the case $\mathfrak{g} = sl_2$, this result has been obtained in [9].

Proposition 4.4. *Let $\Lambda \in P_+^k$ and $\lambda \in \Lambda + Q$. Then $M^{\Lambda, \lambda + k\beta}$ and $M^{\Lambda, \lambda}$ are isomorphic for any $\beta \in Q_L$.*

Proof. (1) follows from the decompositions (4.1)-(4.3). \square

We next investigate more connection between different $M^{\Lambda, \lambda}$ and $M^{\Lambda', \lambda'}$. For this purpose, we need to discuss the simple currents for the vertex operator algebra $L_{\hat{\mathfrak{g}}}(k, 0)$ following [41].

Let $\Lambda_1, \dots, \Lambda_l$ be the fundamental weights of \mathfrak{g} . Then $P = \oplus_{i=1}^l \mathbb{Z}\Lambda_i$ is the weight lattice. Let $\theta = \sum_{i=1}^l a_i \alpha_i$. Here is a list of $a_i = 1$ using the labeling from [30]:

$$\begin{aligned} A_l : & \quad a_1, \dots, a_l \\ B_l : & \quad a_1 \\ C_l : & \quad a_l \\ D_l : & \quad a_1, a_{l-1}, a_l \\ E_6 : & \quad a_1, a_5 \\ E_7 : & \quad a_6 \end{aligned}$$

There are $|P/Q| - 1$ such i with $a_i = 1$ [41].

It is proved in [39] and [41] that $L_{\hat{\mathfrak{g}}}(k, k\Lambda_i)$ are simple current if $a_i = 1$. To see this we let $h^i \in \mathfrak{h}$ for $i = 1, \dots, l$ defined by $\alpha_i(h^j) = \delta_{i,j}$ for $j = 1, \dots, l$. For any $h \in \mathfrak{h}$ set

$$\Delta(h, z) = z^{h(0)} \exp \left(\sum_{n=1}^{\infty} \frac{h(n)(-z)^{-n}}{-n} \right).$$

The following result was obtained in [39] and [41].

Theorem 4.5. *Assume $a_i = 1$.*

(1) *For any $\Lambda \in P_+^k$, $L_{\hat{\mathfrak{g}}}(k, \Lambda)^{(h^i)} = (L_{\hat{\mathfrak{g}}}(k, \Lambda)^{(h^i)}, Y_i)$ is an irreducible $L_{\hat{\mathfrak{g}}}(k, 0)$ -module where $L_{\hat{\mathfrak{g}}}(k, \Lambda)^{(h^i)} = L_{\hat{\mathfrak{g}}}(k, \Lambda)$ as vector spaces and $Y_i(u, z) = Y(\Delta(h^i, z)u, z)$ for $u \in L_{\hat{\mathfrak{g}}}(k, 0)$. Let $\Lambda^{(i)} \in P_+^k$ such that $L_{\hat{\mathfrak{g}}}(k, \Lambda)^{(h^i)}$ is isomorphic to $L_{\hat{\mathfrak{g}}}(k, \Lambda^{(i)})$.*

(2) *The $L_{\hat{\mathfrak{g}}}(k, 0^{(i)}) = L_{\hat{\mathfrak{g}}}(k, k\Lambda_i)$ is a simple current and $L_{\hat{\mathfrak{g}}}(k, k\Lambda_i) \boxtimes L_{\hat{\mathfrak{g}}}(k, \Lambda) = L_{\hat{\mathfrak{g}}}(k, \Lambda^{(i)})$ for all $\Lambda \in P_+^k$.*

Although we do not need to know $\Lambda^{(i)}$ explicitly in this paper, it is still an interesting problem to find out. In the case $\mathfrak{g} = sl_2$, assume $\Delta = \{\pm\alpha\}$. Then $P_+^k = \{\frac{s\alpha}{2} | s = 0, \dots, k\}$ and $(\frac{s\alpha}{2})^{(1)} = \frac{(k-s)\alpha}{2}$ where 1 corresponds to h^1 .

Recall decomposition (4.1). We now investigate how \mathfrak{h} acts on each weight space $L_{\hat{\mathfrak{g}}}(k, \Lambda)(\lambda)$ regarding as a subspace of $L_{\hat{\mathfrak{g}}}(k, \Lambda)^{(h^i)}$. This result will be helpful in the identification of irreducible $K(\mathfrak{g}, k)$ -modules later.

Note that for $h \in \mathfrak{h}$, $\Delta(h^i, z)h(-1)\mathbf{1} = h(-1)\mathbf{1} + k\langle h^i, h \rangle z^{-1}$. Thus

$$Y_i(h(-1)\mathbf{1}, z) = Y(h(-1)\mathbf{1}, z) + k\langle h^i, h \rangle z^{-1}.$$

In particular, the $h(0)$ acts $(L_{\hat{\mathfrak{g}}}(k, \Lambda), Y_i)$ as $h(0) + \langle h^i, h \rangle k$ and acts on $L_{\hat{\mathfrak{g}}}(k, \Lambda)(\lambda) \subset L_{\hat{\mathfrak{g}}}(k, \Lambda)^{(h^i)}$, as $\lambda(h) + \langle h^i, h \rangle k$ for $\lambda \in \Lambda + Q$. Recall the identification between \mathfrak{h} and \mathfrak{h}^* .

We see that $h^i = \frac{2t_{\Lambda_i}}{\langle \alpha_i, \alpha_i \rangle}$. Note from [30] that if $a_i = 1$ then α_i is a long root. So $h^i = t_{\Lambda_i}$ with $a_i = 1$. This implies that $\langle h^i, h \rangle = \Lambda_i(h)$ and

$$\lambda(h) + \langle h^i, h \rangle k = \lambda(h) + k\Lambda_i(h).$$

Thus we have proved the following result:

Lemma 4.6. *There is an $L_{\widehat{\mathfrak{g}}}(k, 0)$ -module isomorphism*

$$f_{\Lambda, i} : L_{\widehat{\mathfrak{g}}}(k, \Lambda)^{(h^i)} \rightarrow L_{\widehat{\mathfrak{g}}}(k, \Lambda^{(i)})$$

such that $f_{\Lambda, i}(L_{\widehat{\mathfrak{g}}}(k, \Lambda)(\lambda)) = L_{\widehat{\mathfrak{g}}}(k, \Lambda^{(i)})(\lambda + k\Lambda_i)$ for all $\lambda \in \Lambda + Q$.

It is easy to see that $\Delta(h^i, z)u = u$ for $u \in K(\mathfrak{g}, k)$. Thus $Y_i(u, z) = Y(u, z)$ for $u \in K(\mathfrak{g}, k)$ and $f_{\Lambda, i} : L_{\widehat{\mathfrak{g}}}(k, \Lambda) \rightarrow L_{\widehat{\mathfrak{g}}}(k, \Lambda^{(i)})$ is a $K(\mathfrak{g}, k)$ -module isomorphism. This gives us the second identification.

Theorem 4.7. *We have a $K(\mathfrak{g}, k)$ -module isomorphism between $M^{\Lambda, \lambda}$ and $M^{\Lambda^{(i)}, \lambda + k\Lambda_i}$ for any $\lambda \in \Lambda + Q$. Moreover, Λ_i does not lie in Q_L . That is, this identification is different from the identification given in Proposition 4.4.*

Proof. The identification between $M^{\Lambda, \lambda}$ and $M^{\Lambda^{(i)}, \lambda + k\Lambda_i}$ for $\lambda \in \Lambda + Q$ is an immediate consequence of Lemma 4.6.

To prove the identification here is different from that given in Proposition 4.4, it is sufficient to show that Λ_i does not lie in Q_L . This is clear if \mathfrak{g} is of A, D, E type as $P = Q \cup \cup_{i, a_i=1} (Q + \Lambda_i)$. So we only need to deal with type B_l and C_l .

For type B_l , let $\mathbb{E} = \mathbb{R}^l$ with the standard orthonormal basis $\{\epsilon_1, \dots, \epsilon_l\}$. Then

$$\Delta = \{\pm\epsilon_i, \pm(\epsilon_i \pm \epsilon_j) | i \neq j\}$$

and $\alpha_1 = \epsilon_1 - \epsilon_2, \dots, \alpha_{l-1} = \epsilon_{l-1} - \epsilon_l, \alpha_l = \epsilon_l$. In this case only $a_1 = 1$. Obviously, $\Lambda_1 = \epsilon_1$ is not an element of Q_L .

For type C_l , let \mathbb{E} be the same as before. Then

$$\Delta = \{\pm\sqrt{2}\epsilon_i, \pm\frac{1}{\sqrt{2}}(\epsilon_i \pm \epsilon_j) | i \neq j\}$$

and $\alpha_1 = \frac{\epsilon_1 - \epsilon_2}{\sqrt{2}}, \dots, \alpha_{l-1} = \frac{\epsilon_{l-1} - \epsilon_l}{\sqrt{2}}, \alpha_l = \sqrt{2}\epsilon_l$. We know only $a_l = 1$. Then

$$\Lambda_l = \frac{\epsilon_1 + \dots + \epsilon_l}{\sqrt{2}}$$

and $Q_L = \sum_{i=1}^l \mathbb{Z}\sqrt{2}\epsilon_i$. It is easy to see that Λ_l does not belong to Q_L . \square

5 Representations of $K(\mathfrak{g}, k)$

In this section we establish the rationality for vertex operator algebra for $K(\mathfrak{g}, k)$ and determine the irreducible modules for $K(\mathfrak{g}, k)$.

Theorem 5.1. *Let \mathfrak{g} be any finite dimensional simple Lie algebra and k be any positive integer. Then*

(1) *The parafermion vertex operator algebra $K(\mathfrak{g}, k)$ is rational.*

(2) *Any irreducible $K(\mathfrak{g}, k)$ -module is isomorphic to $M^{\Lambda, \lambda}$ for some $\Lambda \in P_+^k$ and $\lambda \in \Lambda + Q$. Moreover $M^{\Lambda, \lambda}$ and $M^{\Lambda, \lambda + k\beta}$ are isomorphic for any $\beta \in Q_L$, and $M^{\Lambda, \lambda}$ and $M^{\Lambda^{(i)}, \lambda + k\Lambda_i}$ are isomorphic for any i with $a_i = 1$.*

Proof. (1) Let G be the dual group of the abelian group Q/kQ_L . Then G is a finite subgroup of automorphisms of $L_{\hat{\mathfrak{g}}}(k, 0)$ such that $g \in G$ acts as $g(\beta_i + kQ_L)$ on $V_{\sqrt{k}Q_L + \frac{1}{\sqrt{k}}\beta_i} \otimes M^{0, \beta_i}$. In other words, each $\beta_i + kQ_L$ is an irreducible character of G . So $V_{\sqrt{k}Q_L + \frac{1}{\sqrt{k}}\beta_i} \otimes M^{0, \beta_i}$ in the decomposition

$$L_{\hat{\mathfrak{g}}}(k, 0) = \bigoplus_{i \in Q/kQ_L} V_{\sqrt{k}Q_L + \frac{1}{\sqrt{k}}\beta_i} \otimes M^{0, \beta_i}$$

corresponds to the character $\beta_i + kQ_L$. In particular, $L_{\hat{\mathfrak{g}}}(k, 0)^G = V_{\sqrt{k}Q_L} \otimes K(\mathfrak{g}, k)$.

Since G is a finite abelian group, it follows from Theorem 1 of [42], Theorem 5.24 of [6], $V_{\sqrt{k}Q_L} \otimes K(\mathfrak{g}, k)$ is rational and C_2 -cofinite. As $V_{\sqrt{k}Q_L}$ is rational, the rationality of $K(\mathfrak{g}, k)$ follows immediately.

(2) Since $L_{\hat{\mathfrak{g}}}(k, 0)$, $V_{\sqrt{k}Q_L}$, $K(\mathfrak{g}, k)$ are rational, C_2 -cofinite, CFT type, and $K(\mathfrak{g}, k)$, $V_{\sqrt{k}Q_L}$ are commutants each other in $L_{\hat{\mathfrak{g}}}(k, 0)$, we know from Theorem [31] that every irreducible $K(\mathfrak{g}, k)$ -module occurs in an irreducible $L_{\hat{\mathfrak{g}}}(k, 0)$ -module. The rest follows from Proposition 4.4 and Theorem 4.7. \square

A complete identification of irreducible $K(\mathfrak{g}, k)$ -modules is given in [1] by using the quantum dimensions and global dimensions.

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